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Addendum

Addendum to “Spacelike hypersurfaces with constant higher order mean curvature in the Minkowski space–time”☆

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1. Introduction

In [1], the second and third authors developed a series of general integral formulas for compact spacelike hypersurfaces with hyperplanar boundary in the $(n+1)$-dimensional Minkowski space–time $\mathbb{L}^{n+1}$. As an application of those integral formulas, they proved that hyperplanar balls and hyperbolic caps are the only compact spacelike hypersurfaces in the Minkowski space–time with constant higher order mean curvature and spherical boundary. In this appendix, we would like to emphasize the interest and usefulness of those integral formulas by extending those results in [1] to the case where two (non-necessarily constant) higher order mean curvatures are linearly related. Specifically, we will prove the following uniqueness result (recall that $H_0 = 1$ by definition).

☆ Preceding paper.

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Theorem 1. *Let $x : M^n \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurfacein the Minkowski space–time with spherical boundary. Assume that for integers $0 \leq \ell < k \leq n$, the higher order mean curvatures H_ℓ and H_k are linearly related by $H_k = cH_\ell$, for a constant c. Then M is either a hyperplanar ball or a hyperbolic cap.*

As stated in its own title, this note should be read as an addendum to the paper [1]. For that reason, we will directly follow the notation and nomenclature in [1].

2. Proof of the theorem

Since $H_0 = 1$ by definition, the case $\ell = 0$ corresponds to Theorems 1 and 2 in [1], so that we may assume without loss of generality that $\ell \geq 1$ (and hence $k \geq 2$). Following the notation in Lemma 2 [1], let us assume that the boundary is contained in the hyperplane $\Pi = a^\perp$. Then, we know from the integral formula (22) in [1] that

$$\oint_{\partial M} h_{k-1}\langle \nu, a\rangle^k \,\mathrm{d}s = n\int_M H_k\langle a, N\rangle \,\mathrm{d}M = nc\int_M H_\ell\langle a, N\rangle \,\mathrm{d}M,$$

and

$$\oint_{\partial M} h_{\ell-1}\langle \nu, a\rangle^\ell \,\mathrm{d}s = n\int_M H_\ell\langle a, N\rangle \,\mathrm{d}M,$$

that is,

$$\oint_{\partial M} h_{k-1}\langle \nu, a\rangle^k \,\mathrm{d}s = c\oint_{\partial M} h_{\ell-1}\langle \nu, a\rangle^\ell \,\mathrm{d}s. \tag{2.1}$$

Suppose now the boundary $\Sigma = x(\partial M)$ is a round sphere $\mathbb{S}^{n-1}(\rho)$ of radius $\rho > 0$. In that case, we have that $\tau_i = -1/\rho$ for every $i = 1, \ldots, n-1$, so that $h_{r-1} = (-1)^{r-1}/\rho^{r-1}$ for $r = 1, \ldots, n-1$. Besides, $\mathrm{vol}(D) = \rho A_\rho/n$, where $A_\rho = \mathrm{area}(\mathbb{S}^{n-1}(\rho))$. Therefore, Eq. (2.1) becomes

$$\oint_{\partial M} \langle \nu, a\rangle^k \,\mathrm{d}s = (-1)^{k-\ell}c\rho^{k-\ell}\oint_{\partial M} \langle \nu, a\rangle^\ell \,\mathrm{d}s. \tag{2.2}$$

We may assume without loss of generality that there exists at least an elliptic point $p_0 \in M$, where $H_r(p_0) > 0$ for every r. Otherwise, we know from Lemma 1 in [1] that the hypersurface is a hyperplanar ball. We claim that $H_k(p) > 0$ for every point $p \in M$. Indeed, observe that c must be a positive constant, because $c = H_k(p_0)/H_\ell(p_0) > 0$ at an elliptic point p_0. Denote

$$U = \{p \in M : H_k(p) > 0\}.$$

It is clear that U is a non-empty open subset of M, since $p_0 \in U$. We will show that it is also closed. By Gårding inequalities [2,3] (taking into account the sign convention in the definition of H_r [1]), we know that at each point $p \in U$

$$H_\ell^{k/\ell}(p) \geq H_k(p) = cH_\ell(p) > 0,$$

that is,

$$H_\ell(p) \geq c^{\ell/(k-\ell)} > 0,$$

at each point $p \in U$. This gives

$$H_k(p) = cH_\ell(p) \geq c^{k/(k-\ell)} > 0, \quad \text{for every } p \in U,$$

showing that $U = \{p \in M : H_k(p) \geq c^{k/(k-\ell)} > 0\}$ is also closed.

Therefore, $M = U$ and $H_k > 0$ on the whole M, as we claimed. Then Gårding inequalities imply that

$$H_1 \geq H_2^{1/2} \geq \cdots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} > 0, \tag{2.3}$$

and

$$\frac{H_1}{H_0} \geq \frac{H_2}{H_1} \geq \cdots \geq \frac{H_k}{H_{k-1}}, \tag{2.4}$$

hold on M, with equality at any stage only at umbilical points. Since $k \geq 2$, taking $r = 2$ in Eq. (16) in [1], we can conclude from the fact that $H_2 > 0$ on M that $\langle \nu, a \rangle$ cannot vanish at any boundary point $p \in \partial M$. Otherwise, we would obtain at that point that

$$0 < \binom{n}{2} H_2(p) = -\sum_{i=1}^{n-1} \langle A\nu, e_i \rangle^2 \leq 0,$$

which is not possible. Besides, we also know from (22) in [1] that

$$\oint_{\partial M} \langle \nu, a \rangle \, \mathrm{d}s = n \int_M H_1 \langle a, N \rangle \, \mathrm{d}M. \tag{2.5}$$

Therefore, taking into account that $H_1 > 0$ and $\langle a, N \rangle < 0$ on M, it follows from here that $\langle \nu, a \rangle < 0$ on ∂M. This allows us to rewrite (2.2) as

$$\oint_{\partial M} |\langle \nu, a \rangle|^\ell \, \mathrm{d}s = \frac{1}{c\rho^{k-\ell}} \oint_{\partial M} |\langle \nu, a \rangle|^k \, \mathrm{d}s. \tag{2.6}$$

By the Holder inequality, we obtain from here that

$$\oint_{\partial M} |\langle \nu, a \rangle|^\ell \, \mathrm{d}s \leq \left(\oint_{\partial M} |\langle \nu, a \rangle|^k \, \mathrm{d}s \right)^{\ell/k} A_\rho^{(k-\ell)/k}$$

which jointly with (2.6) gives

$$\oint_{\partial M} |\langle \nu, a \rangle|^k \, \mathrm{d}s \leq c^{k/(k-\ell)} \rho^k A_\rho. \tag{2.7}$$

Finally, by the Holder inequality we also get that

$$\oint_{\partial M} |\langle \nu, a\rangle|\, \mathrm{d}s \le \left(\oint_{\partial M} |\langle \nu, a\rangle|^k\, \mathrm{d}s\right)^{1/k} A_\rho^{(k-1)/k},$$

which along with (2.7) implies the following inequality:

$$\left|\oint_{\partial M} \langle \nu, a\rangle\, \mathrm{d}s\right| = \oint_{\partial M} |\langle \nu, a\rangle|\, \mathrm{d}s \le c^{1/(k-\ell)} \rho A_\rho. \tag{2.8}$$

This corresponds to inequality (25) in the proof of Theorem 1 in [1].

On the other hand, from (2.4), we deduce that

$$c = \frac{H_k}{H_\ell} \le \frac{H_{k-\ell}}{H_{\ell-\ell}} = H_{k-\ell},$$

so that by (2.3) it follows that

$$H_1 \ge H_{k-\ell}^{1/(k-\ell)} \ge c^{1/(k-\ell)},$$

with equality only at umbilical points. Therefore, as in the proof of Theorem 1 in [1], we have

$$nH_1(-\langle a, N\rangle) \ge nc^{1/(k-\ell)}(-\langle a, N\rangle) > 0,$$

with equality if and only if M is totally umbilical. Then, integrating this inequality on M and using (2.5) (along with (14) in [1]), we deduce that

$$\begin{aligned}\left|\oint_{\partial M} \langle \nu, a\rangle\, \mathrm{d}s\right| &= \oint_{\partial M} |\langle \nu, a\rangle|\, \mathrm{d}s = n\int_M H_1(-\langle a, N\rangle) \ge nc^{1/(k-\ell)} \int_M (-\langle a, N\rangle) \\ &= nc^{1/(k-\ell)}\, \mathrm{vol}(D) = \rho c^{1/(k-\ell)} A_\rho, \end{aligned} \tag{2.9}$$

with equality if and only if M is totally umbilical. This corresponds to inequality (27) in [1]. Finally, by (2.8), we have the equality in (2.9) and then M must be umbilical. This finishes the proof of our result.

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